

# Quadratic Poisson brackets and Drinfeld theory for associative algebras

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## Abstract

The paper is devoted to the Poisson brackets compatible with multiplication in associative algebras. These brackets are shown to be quadratic and their relations with the classical Yang–Baxter equation are revealed. The paper also contains a description of Poisson Lie structures on Lie groups whose Lie algebras are adjacent to an associative structure.

## 1 Introduction

In his famous work [1] V.G.Drinfeld found remarkable relations between skew-symmetric solutions of classical Yang–Baxter equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \quad (1)$$

where  $r$  is an element of the tensor square of some Lie algebra  $\mathfrak{g}$ , and Poisson Lie structures on the corresponding Lie group  $G$ . More exactly, fix some basis  $e_i$  in  $\mathfrak{g}$ , and let  $r = r^{ij}e_i \wedge e_j$  be a solution of (1). Let us identify  $\mathfrak{g}$  with the tangent space of  $G$  in the unit, and let  $E_i$  and  $E'_i$  be left- and right-invariant vector fields on  $G$  extending the basis  $e_i$ . The Poisson Lie tensor (see below exact definitions, as well as explanation of notation) at the point  $x \in G$  is then given by

$$\pi^{ij}(x) = r^{ij}(E_i(x) \otimes E_j(x) - E'_i(x) \otimes E'_j(x)). \quad (2)$$

Formula (2) has many interesting consequences (see [2] for some examples). But computations involving it necessarily include direct formulas for  $E_i(x)$  and  $E'_i(x)$  and may be very cumbersome. Things improve, however, in case Lie algebra  $\mathfrak{g}$  comes from an *associative* one.

If  $A$  is an associative algebra, then one may define a Lie algebra  $A_L$  taking  $[a, b] = ab - ba$ . If  $A$  has a unit, then the group  $G$  of invertible elements of algebra  $A$  is a Lie group whose Lie algebra is  $A_L$ . So, to obtain Poisson Lie structures on  $G$ , one can start from Poisson brackets compatible with the algebra  $A$ . These brackets are relatively easy to study because algebra  $A$  is topologically trivial (a linear space). It turns out to be

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that, under some conditions of smoothness, all these brackets are quadratic. Quadratic Poisson brackets were studied by many authors (see e.g. a series of works [3, 4, 5] by B.Kupershmidt who considered an important case of a full matrix algebra, and also M.A.Semenov-Tyan-Shanskii's [6]). Quadratic brackets compatible with an associative algebra reveal some new relations to the classical Yang–Baxter equation which suggest a possibility to obtain their quantization. This can be done using quadratic algebras (see [7]) and will be a subject of the forthcoming paper. Quantization of the corresponding Poisson Lie groups may also be simpler than in general case (see e.g. [8]) because such Poisson Lie groups bear a preferred coordinate system (linear functions on the original algebra).

The paper is structured as follows. Section 2 deals with quadratic brackets compatible with an associative algebra structure of  $A$ . It is shown that these brackets correspond to *differentiations* of the algebra  $\text{Sym}(A \otimes A)$  (symmetric elements of  $A \otimes A$ ) with values in  $A \wedge A$ . Jacobi identity means that this differentiation satisfies some version of classical Yang–Baxter equation. Then, it is shown in Section 3 that restriction of the brackets considered in Section 2 to the group of invertible elements gives rise to a Poisson Lie structure. It is also proved that any coboundary Poisson Lie structure on a simply connected Lie group is a Poisson covering over a Poisson Lie group whose Poisson bracket is quadratic in some global coordinate system. Section 4 contains examples.

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## Definitions and notation

Throughout this paper  $A$  stands for a finite-dimensional associative algebra,  $e_i$  ( $i = 1, \dots, n$ ) is its (additive) basis, and  $a_{ij}^k$  are structure constants of the algebra with respect to the basis  $e_i$ :

$$e_i e_j = a_{ij}^k e_k \quad (3)$$

(a summation over repeated indices will be always assumed). The symbol  $A_L$  will denote an *adjacent* Lie algebra, i.e. Lie algebra on the same space with a commutator  $[a, b] = ab - ba$ .

As usual, *Poisson bracket*  $\{\cdot, \cdot\}$  on a smooth manifold  $M$  is understood as a Lie algebra structure on the space of smooth functions  $C^\infty(M)$  satisfying the Leibnitz identity, i.e. a bilinear operation  $\{\cdot, \cdot\}$  such that

1.  $\{f, g\} = -\{g, f\}$ ,
2.  $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$  (Jacobi identity), and
3.  $\{fg, h\} = f\{g, h\} + \{f, h\}g$  (Leibnitz identity).

If Jacobi identity is not required one speaks about pre-Poisson brackets. One can easily see that in local coordinates  $x^i$  on the manifold  $M$  an arbitrary pre-Poisson bracket

looks like

$$\{f, g\}(x) = \pi^{ij}(x) \partial f(x) / \partial x^i \cdot \partial g(x) / \partial x^j, \quad (4)$$

for some *Poisson tensor*  $\pi^{ij}(x)$ . Bracket (4) is Poisson if and only if the Poisson tensor satisfies the equation:

$$\pi^{lk}(x) \frac{\partial \pi^{ij}(x)}{\partial x^l} + \pi^{li}(x) \frac{\partial \pi^{jk}(x)}{\partial x^l} + \pi^{lj}(x) \frac{\partial \pi^{ki}(x)}{\partial x^l} = 0. \quad (5)$$

The set of all Poisson brackets on a given manifold is not a linear space but just a cone. Indeed, a sum of two Poisson brackets is pre-Poisson, but it usually does not satisfy Jacobi identity. We call two brackets  $\{\cdot, \cdot\}_1$  and  $\{\cdot, \cdot\}_2$  *compatible* with one another if any linear combination  $\alpha\{\cdot, \cdot\}_1 + \beta\{\cdot, \cdot\}_2$  is again a Poisson bracket.

A product  $M_1 \times M_2$  of two manifolds equipped with Poisson brackets  $\{\cdot, \cdot\}_1$  and  $\{\cdot, \cdot\}_2$ , respectively, may be given a *product* Poisson bracket. The latter is a Poisson bracket whose Poisson tensor  $\pi_p^{ij}(x, y)$  in the point  $(x, y) \in M_1 \times M_2$  is

$$\pi_p(x, y) = \begin{pmatrix} \pi_1^{ij}(x) & 0 \\ 0 & \pi_2^{ij}(y) \end{pmatrix} \quad (6)$$

where  $\pi_1^{ij}$  and  $\pi_2^{ij}$  are Poisson tensors of  $\{\cdot, \cdot\}_1$  and  $\{\cdot, \cdot\}_2$ , respectively. If  $p_1 : M_1 \times M_2 \rightarrow M_1$  and  $p_2 : M_1 \times M_2 \rightarrow M_2$  are natural projections, and  $p_1^* : C^\infty(M_1) \rightarrow C^\infty(M_1 \times M_2)$  and  $p_2^* : C^\infty(M_2) \rightarrow C^\infty(M_1 \times M_2)$  are corresponding pullbacks, then it is easy to see that product Poisson bracket may be defined as the only Poisson bracket on  $M_1 \times M_2$  satisfying the conditions:

1.  $\{p_1^*(f), p_1^*(g)\} = \{f, g\}_1$ , and  $\{p_2^*(f), p_2^*(g)\} = \{f, g\}_2$ ,
2.  $\{p_1^*(f), p_2^*(g)\} = 0$ .

A mapping  $F : M_1 \rightarrow M_2$  of two manifolds equipped with Poisson brackets  $\{\cdot, \cdot\}_1$  and  $\{\cdot, \cdot\}_2$ , respectively, is called *Poisson* if a pullback  $F^* : C^\infty(M_2) \rightarrow C^\infty(M_1)$  is a Lie algebra homomorphism, i.e.

$$\{f \circ F, g \circ F\}_1 = \{f, g\}_2 \circ F.$$

Let  $M_1$  be a manifold with a Poisson bracket  $\{\cdot, \cdot\}$ , and  $M' \subset M$  be its submanifold. Then  $M'$  is called *Poisson submanifold* if it carries a Poisson bracket  $\{\cdot, \cdot\}'$  such that the inclusion  $\iota : M' \hookrightarrow M$  is a Poisson mapping. It is easy to see that if the bracket  $\{\cdot, \cdot\}'$  exists, then it is unique. We call it a *restriction* of the Poisson bracket  $\{\cdot, \cdot\}$  to the submanifold  $M'$ .

Let now manifold  $M$  be a Lie group, i.e. be equipped with a smooth multiplication  $*$  :  $M \times M \rightarrow M$ . A Poisson bracket on  $M$  is called *compatible* with this multiplication if  $*$  is a Poisson mapping (with respect to a product Poisson structure in  $M \times M$ ). In this case  $M$  is called a *Poisson Lie group*. More explicitly compatibility means the following: let  $f, g$  be functions from  $C^\infty(M)$ , and  $r(x) = \{f(x), g(x)\}$  be their Poisson bracket. Let also  $F, G, R$  be functions from  $C^\infty(M \times M)$  given by:

$$\begin{aligned} F(x, y) &= f(x * y), \\ G(x, y) &= g(x * y), \\ R(x, y) &= r(x * y). \end{aligned}$$

Then there should be

$$R = \{F, G\}_p \quad (7)$$

where  $\{\cdot, \cdot\}_p$  means a product Poisson bracket (6).

## 2 Quadratic Poisson brackets compatible with an algebra structure

Consider now Poisson structures on a linear space  $A$ . The dual space  $A^*$  is naturally embedded into functions algebra  $C^\infty(A)$ , as well as all its symmetric powers  $\text{Sym}((A^*)^{\otimes n})$  are. In view of (4) speaking about Poisson structures we may (and will) even identify  $C^\infty(A)$  with the full symmetric algebra  $\oplus_{n=0}^\infty \text{Sym}((A^*)^{\otimes n})$ . The algebra  $C^\infty(A \times A)$  is then identified with  $C^\infty(A) \otimes C^\infty(A)$ .

**Definition** We say that Poisson bracket  $\delta^* : C^\infty(A) \wedge C^\infty(A) \rightarrow C^\infty(A)$  is a bracket of degree  $N$  if

$$\delta^*(A^* \wedge A^*) \subset \text{Sym}((A^*)^{\otimes N}).$$

In other words, in natural local coordinates  $x^i$  on the space  $A$  the Poisson tensor  $\pi^{ij}(x^1, \dots, x^n)$  of the bracket  $\delta^*$  must be a homogeneous polynomial of degree  $N$ . It is easy to see that the values of bracket  $\delta^*$  on the space  $A^* \wedge A^*$  (i.e. on linear functions) uniquely determine, by Leibnitz identity, its values on the whole  $C^\infty(A) \wedge C^\infty(A)$ , so that we will denote a restriction of  $\delta^*$  to  $A^* \wedge A^*$  by the same symbol. We chose an unusual name  $\delta^*$  for the bracket because we will soon make an active use of its dual.

Brackets of degree 1 will be called linear, of degree 2, quadratic, etc. In particular, linear (Berezin–Lie) brackets are just the brackets compatible (in the sense of the previous Section) with addition operation in  $A$ . The next-simplest case is quadratic brackets (see [9], [10], [11]). They play a special role in our considerations due to the following

**Lemma 1** *A Poisson bracket compatible with the multiplication in an algebra  $A$  with a unit is quadratic. Its Poisson tensor in the unit of  $A$  is zero.*

**Proof** Let  $e_i$  be, as usual, a basis of the algebra  $A$ ,  $u = u^i e_i$  be its unit, and  $x^i$  be a dual basis in  $A^*$ . Taking  $f = x^i$ ,  $g = x^j$  in the compatibility condition (7), one obtains the following identity:

$$\pi^{ij}(y * z) = a_{pq}^i a_{st}^j (z^q z^t \pi^{ps}(y) + y^p y^s \pi^{qt}(z)) \quad (8)$$

for any  $y = y^k e_k$  and  $z = z^k e_k$ . We are now to extract information from (8) assigning special values to  $y$  and  $z$ .

First, take  $y = z = tu$  where  $t \in \mathbb{R}$  is arbitrary. This gives:

$$\begin{aligned} \pi^{ij}(t^2 u) &= t^2 a_{pq}^i a_{st}^j (u^q u^t \pi^{ps}(tu) + u^p u^s \pi^{qt}(tu)) \\ &= t^2 \langle x^i, e_p u \rangle \langle x^j, e_s u \rangle \pi^{ps}(tu) + t^2 \langle x^i, u e_q \rangle \langle x^j, u e_t \rangle u^p u^s \pi^{qt}(tu) \\ &= t^2 \delta_p^i \delta_s^j \pi^{ps}(tu) + t^2 \delta_q^i \delta_t^j \pi^{qt}(tu) \\ &= 2t^2 \pi^{ij}(tu). \end{aligned}$$

Denote  $q(t) \stackrel{\text{def}}{=} \pi^{ij}(tu)$ . Thus, the function  $q$  satisfies the equation

$$q(t^2) = 2t^2 q(t). \quad (9)$$

Substitution  $t = 0, 1$  gives:

$$q(0) = q(1) = 0, \quad (10)$$

proving the second assertion of the lemma. Take now some  $\alpha \in \mathbb{R}$ ,  $\alpha > 0$ . The sequence  $\alpha_n \stackrel{\text{def}}{=} \alpha^{1/2^n}$  tends to 1 as  $n \rightarrow \infty$ , so, by continuity,  $q(\alpha_n)$  should also tend to  $0 = q(1)$  (recall that, by definition, the Poisson tensor  $\pi^{ij}(x) = \{x^i, x^j\}$  is supposed to be a smooth function). But (9) implies, by induction in  $n$ , that

$$q(\alpha_n) = \frac{q(\alpha)}{2^n \alpha^{2-1/2^n}}. \quad (11)$$

Now, combining (11) with (10) and the fact that  $q(s)$  is smooth, one obtains the following equation:

$$q'(1) = \lim_{n \rightarrow \infty} \frac{q(\alpha_n)}{1 - \alpha_n} = \frac{q(\alpha)}{\alpha^2 \log \alpha} \quad (12)$$

and, therefore, it should be  $q(\alpha) = C \alpha^2 \log \alpha$  for some constant  $C$  and for all  $\alpha > 0$ . But  $q(t)$  is smooth at  $t = 0$ , so that  $C = 0$ , and  $q(t) \equiv 0$  for all  $t \geq 0$ . But equation (9) shows that function  $q$  is even:  $q(-t) = q(t)$ , and therefore allows to extend the latter identity to all real  $t$ :

$$\pi^{ij}(tu) = q(t) \equiv 0. \quad (13)$$

Take now in (8)  $z = tu$  and  $y \in A$  arbitrary to obtain

$$\begin{aligned} \pi^{ij}(ty) &= t^2 a_{pq}^i a_{st}^j (u^q u^t \pi^{ps}(y) + y^p y^s \pi^{qt}(tu)) \\ &= t^2 \langle x^i, e_p u \rangle \langle x^j, e_s u \rangle \pi^{ps}(y) \\ &= t^2 \delta_p^i \delta_s^j \pi^{ps}(y) = t^2 \pi^{ij}(y). \end{aligned}$$

A smooth function satisfying this identity is necessarily a quadratic form. ■

Notice a key role that *smoothness* of  $\pi^{ij}(x)$  plays in the proof; see Example 4 below.

Thus, being interested in brackets compatible with algebras, we may restrict ourselves to quadratic ones. A general coordinate expression for the quadratic Poisson (and also pre-Poisson) bracket  $\delta^*$  is:

$$\{x^i, x^j\} = c_{kl}^{ij} x^k x^l, \quad (14)$$

where symbol  $c_{kl}^{ij}$  is skew-symmetric with respect to the upper indices,  $c_{kl}^{ij} = -c_{kl}^{ji}$ . As to the lower indices, the question of symmetry is a priori meaningless because only the sums  $c_{kl}^{ij} + c_{lk}^{ij}$  are fixed, and values of individual  $c_{kl}^{ij}$  may be chosen. A dual mapping  $\delta : \text{Sym}(A \otimes A) \rightarrow A \wedge A$  is given by as

$$\delta(e_k \otimes e_l + e_l \otimes e_k) = (c_{kl}^{ij} + c_{lk}^{ij}) e_i \otimes e_j. \quad (15)$$

(linear functions  $x^i \in A^*$  are supposed to be the basis dual to  $e_i \in A$ ).

A tensor square  $A \otimes A$  of the algebra  $A$  can also be given an algebra structure by a componentwise multiplication. Then the set of symmetric tensors  $\text{Symm}(A \otimes A) \subset A \otimes A$  is its subalgebra, while the set of skew-symmetric tensors  $A \wedge A \subset A \otimes A$  is a  $\text{Symm}(A \otimes A)$ -bimodule. A linear mapping  $D : \mathfrak{A} \rightarrow V$  from algebra  $\mathfrak{A}$  to a  $\mathfrak{A}$ -bimodule  $V$  is called a *differentiation* if it satisfies the condition

$$D(p \cdot q) = pD(q) + D(p)q \quad (16)$$

for all  $p, q \in \mathfrak{A}$ .

**Theorem 1** *Quadratic Poisson bracket  $\delta^* : A^* \wedge A^* \rightarrow \text{Symm}(A^* \otimes A^*)$  is compatible with the algebra structure in  $A$  if and only if its dual mapping  $\delta$  is a differentiation of  $\text{Symm}(A \otimes A)$  with values in  $A \wedge A$ .*

The proof is just a computation: one has to check that condition (16) for the mapping (15) and condition of compatibility of the mapping (14) with multiplication (3) give rise to the same identities. The details are carried out in [10]. The proof does not use Jacobi identity and therefore applies for general pre-Poisson brackets as well.

Let  $V$  be a linear space and  $P$  be a linear operator acting from  $V \otimes V$  to itself. Then  $P^{12}$  will mean an operator  $V \otimes V \otimes V \rightarrow V \otimes V \otimes V$  acting as  $P$  on the first and second tensor component, and as the identity operator on the third one. Notations  $P^{23}$  and  $P^{13}$  have similar meaning. *Schouten bracket*  $[[P, P]]$  is then defined as

$$[P^{12}, P^{13}] + [P^{12}, P^{23}] + [P^{13}, P^{23}]. \quad (17)$$

with  $[\cdot, \cdot]$  meaning an ordinary commutator of operators.

Let  $\delta^* : A^* \wedge A^* \rightarrow \text{Symm}(A^* \otimes A^*)$  be a quadratic pre-Poisson bracket,  $\delta : \text{Symm}(A \otimes A) \rightarrow A \wedge A$  be its dual, and  $\tilde{\delta} : A \otimes A \rightarrow A \otimes A$  be an arbitrary linear extension of  $\delta$ . Choosing this extension means simply that we fix coefficients  $c_{kl}^{ij}$  in (14) while a priori only sums  $c_{kl}^{ij} + c_{lk}^{ij}$  are fixed.

**Theorem 2** *Bracket  $\delta^*$  is Poisson (i.e. satisfies Jacobi identity) if and only if*

$$[[\tilde{\delta}, \tilde{\delta}]](X) = 0 \quad (18)$$

for any fully symmetric tensor  $X \in A \otimes A \otimes A$ .

**Remark** In particular, it is asserted in Theorem that  $[[\tilde{\delta}, \tilde{\delta}]](X)$  for fully symmetric  $X$  depends on  $\delta$  only, and not on its specific extension  $\tilde{\delta}$ .

The proof is, again, a straightforward (though rather tedious) computation: one has to check that equation (18) applied to an arbitrary fully symmetric tensor  $X = X^{pqr}e_p \otimes e_q \otimes e_r$  means just the same as Jacobi identity for the bracket (14) applied to an arbitrary triple of linear functions  $x^s, x^u, x^v$ . Then, observe that due to Leibnitz identity the Jacobi identity is satisfied for arbitrary functions if and only if it is true for linear ones. By the way, the computation does not require  $A$  to have a unit.

Note also that it is not assumed in Theorem 2 that bracket  $\delta^*$  is compatible with multiplication in  $A$ , that is,  $\delta$  is a differentiation. Assume however that  $\delta$  is a differentiation, and even has a form:

$$\delta(x) = \text{ad}_r(x) \stackrel{\text{def}}{=} [r, x] \quad (19)$$

for some  $r = r^{ij}e_i \otimes e_j \in A \wedge A$  (it is easy to see that all the mappings (19) are differentiations  $\text{Symm}(A \otimes A) \rightarrow A \wedge A$ ). Suppose that  $A$  has a unit  $u$ , then the following notation is standard:  $r^{12}$ ,  $r^{23}$ , and  $r^{13}$  mean elements of the tensor cube  $A \otimes A \otimes A$  equal to  $r^{ij}e_i \otimes e_j \otimes u$ ,  $r^{ij}u \otimes e_i \otimes e_j$ , and  $r^{ij}e_i \otimes u \otimes e_j$ , respectively. Schouten bracket  $[[r, r]]$  is still defined by expression (17) with  $[\cdot, \cdot]$  meaning now a commutator of elements of  $A \otimes A \otimes A$  rather than operators. Substitution of (19) into (18) leads to the following theorem:

**Theorem 3** *Differentiation  $\delta$  satisfies condition (18) (and therefore defines a Poisson bracket) if and only if  $[[r, r]]$  is ad-invariant, i.e. commutes with any tensor of the type  $S_a = u \otimes u \otimes a + u \otimes a \otimes u + a \otimes u \otimes u$ ,  $a \in A$ .*

**Proof** It is easily checked that  $\text{ad}_{[a, b]} = [\text{ad}_a, \text{ad}_b]$  and therefore  $[[\text{ad}_r, \text{ad}_r]] = \text{ad}_{[[r, r]]}$ . Thus Theorem 2 means that, as soon as  $\delta = \text{ad}_r$  satisfies Jacobi identity,  $\text{ad}_{[[r, r]]}(X) = 0$  for any fully symmetric  $X \in A \otimes A \otimes A$ , and thus, for any tensor of the type  $S_a$ ,  $a \in A$ . Prove now the converse. If an element  $[[r, r]]$  commutes with two elements  $x$  and  $y \in A \otimes A \otimes A$ , then it also commutes with their sum and their product. So, if it commutes with all the elements  $S_x$ , it also commutes with any element

$$\begin{aligned} a \otimes b \otimes c + b \otimes c \otimes a + c \otimes a \otimes b + a \otimes c \otimes b + c \otimes b \otimes a + a \otimes b \otimes c \\ = S_a S_b S_c - S_{ab} S_c - S_{ac} S_b - S_{bc} S_a + S_{acb} + S_{bca}, \end{aligned}$$

and therefore, by linearity, with any fully symmetric tensor  $X \in A \otimes A \otimes A$ . ■

**Corollary 1** *Let  $A$  be associative algebra, and  $\mathfrak{G}$  be a Lie subalgebra of  $A_L$ . Then any element  $r \in \mathfrak{G} \wedge \mathfrak{G}$  satisfying classical Yang–Baxter equation (1) defines a quadratic Poisson bracket compatible with  $A$ .*

### 3 Quadratic brackets and Poisson Lie groups

**Definition** *Lie bialgebra* is a Lie algebra  $\mathfrak{G}$  together with a linear mapping  $\Delta : \mathfrak{G} \rightarrow \mathfrak{G} \wedge \mathfrak{G}$  such that:

1. A dual mapping  $\Delta^* : \mathfrak{G}^* \wedge \mathfrak{G}^* \rightarrow \mathfrak{G}^*$  equips  $\mathfrak{G}^*$  with a Lie algebra structure.
2. The following compatibility condition is satisfied:

$$\Delta([a, b]) = \text{ad}_a \Delta(b) - \text{ad}_b \Delta(a) \quad (20)$$

for arbitrary  $a, b \in \mathfrak{G}$ . Here  $\text{ad}$  stands for the adjoint representation of the Lie algebra  $\mathfrak{G}$  in  $\mathfrak{G} \wedge \mathfrak{G}$ .

Drinfeld in [1] proved the following theorem:

**Theorem 4** *Mapping  $\Delta_r(x) = [r, x \otimes 1 + 1 \otimes x]$  where  $r$  is some element of  $\mathfrak{G} \wedge \mathfrak{G}$ ,  $x \in \mathfrak{G}$  and  $1$  is a formal unit, always satisfies (20). This mapping defines a Lie bialgebra structure if and only if  $[[r, r]]$  is ad-invariant.*

The Lie bialgebras with such  $\Delta$  are called *coboundary*.

Let now  $G$  be a Lie group equipped with a Poisson bracket, and  $\mathfrak{G}$  a corresponding Lie algebra identified with the tangent space to the group  $G$  in its unit  $e$ . Take  $a, b \in \mathfrak{G}^*$ , and choose arbitrary functions  $f(x), g(x)$  on  $G$  such that  $df(e) = a, dg(e) = b$ . Define now the mapping  $\Delta^* : \mathfrak{G}^* \wedge \mathfrak{G}^* \rightarrow \mathfrak{G}^*$  by the formula

$$\Delta^*(a \wedge b) = d\{f, g\}(e). \quad (21)$$

The following theorem is due to Drinfeld (see [1]):

**Theorem 5** *The previous definition is sound, i.e.  $\Delta^*(a \wedge b)$  depends on  $a$  and  $b$  only and not on a specific choice of functions  $f$  and  $g$ . The bracket  $\{\cdot, \cdot\}$  is compatible with the multiplication in  $G$  (i.e.  $G$  is a Poisson Lie group) if and only if  $(\mathfrak{G}, \Delta)$  is a Lie bialgebra. This Lie bialgebra determines the corresponding Poisson bracket uniquely.*

Consider now an associative algebra  $A$  with a unit  $u$ , and let  $G$  be a group of its invertible elements.  $G$  is an open submanifold of  $A$ , and therefore, if  $A$  bears a Poisson bracket, then  $G$  is a Poisson submanifold (i.e., bracket can be restricted to  $G$ ). Restriction of quadratic bracket considered in the previous Section, gives

**Theorem 6** *Let  $A$  be an associative algebra with the unit  $u$ , and  $\delta^* : A^* \wedge A^* \rightarrow \text{Sym}(A^* \otimes A^*)$  be a quadratic Poisson bracket compatible with it. The restriction of  $\delta^*$  to the group  $G$  of invertible elements of the algebra equips  $G$  with a Poisson Lie structure. The corresponding Lie bialgebra structure is given by the formula*

$$\Delta(x) = 2\delta(x \otimes u + u \otimes x). \quad (22)$$

**Proof** Properties 1–3 from definition of Poisson bracket are obviously preserved because  $G$  is an open submanifold of  $A$ . Since  $G$  inherits its multiplication from  $A$ , the bracket and the multiplication are still compatible, so that  $G$  is a Poisson Lie group. To obtain bialgebra structure from Drinfeld's construction, let us take functions  $f$  and  $g$  linear. Thus, one has really to compute a differential of the Poisson tensor (14) in the point  $x = u$ , which obviously gives (22). ■

So,  $\Delta^*$  equips  $A^*$  with the structure of Lie algebra. Coordinate expression for Lie bracket in the basis  $x^i \in A^*$  dual to the basis  $e_i \in A$  looks like:

$$\Delta^*(x^i \wedge x^j) = (c_{kl}^{ij} + c_{lk}^{ij})u^k x^l \quad (23)$$

where bracket  $\delta^*$  is given by (14), and  $u = u^k e_k$ .

This formula has an interesting byproduct. Consider a mapping  $\Delta_a : A \rightarrow A \wedge A$  acting as

$$\Delta_a(x) = \delta(x \otimes a + a \otimes x). \quad (24)$$

where  $a$  is an arbitrary element of  $A$ . It turns out to be that  $\Delta_a^* : A^* \wedge A^* \rightarrow A^*$  is always a Lie bracket, i.e. satisfies Jacobi identity. Indeed, take  $a = a^i e_i$ , and write coordinate expressions for the operation  $\Delta_a^*$ . One can easily see that it is just (23) with  $a_i$  substituted for  $u_i$ . We know from Theorem 6 that  $\Delta_u^*$  satisfies Jacobi identity. But if we checked it in coordinates we would not make use of the fact that  $u$  is a unit of algebra, and therefore the proof applies to an arbitrary  $a$  as well. Note also that formula (24) may be viewed as an expression for the Poisson tensor of a *linear* pre-Poisson bracket, and the last assertion means then that this bracket is really Poisson. So, any quadratic bracket compatible with an associative algebra  $A$  equips it also with a series of linear Poisson brackets.

Moreover, the following theorem holds:

**Theorem 7** *Brackets  $\Delta^*$  and  $\Delta_u^*$  (where  $u$  is a unit of algebra  $A$ ) are compatible with one another, i.e. their arbitrary linear combination  $\alpha\Delta^* + \beta\Delta_u^*$  is also a Poisson bracket.*

**Proof** The case  $\alpha = 0$  has already been considered, so take  $\alpha = 1$  and  $\beta = t$ . If the function  $\pi^{ij}(x) = c_{kl}^{ij} x^k x^l$  satisfies (5) (which means that Jacobi identity holds), then so does the function

$$\Pi^{ij}(x) \stackrel{\text{def}}{=} \pi^{ij}(x + tu) = c_{kl}^{ij} x^k x^l + t(c_{kl}^{ij} + c_{lk}^{ij}) u^i x^j + t^2 c_{kl}^{ij} u^k u^l$$

The last term vanishes by Lemma 1, and the rest is exactly the Poisson tensor for  $\Delta^* + t\Delta_u^*$ . ■

Take up again to coboundary Lie bialgebras to prove

**Theorem 8** *Let  $G$  be a connected simply connected Poisson Lie group such that its Lie algebra  $\mathfrak{G} = A_L$  where  $A$  is an associative algebra with the unit. Let the corresponding Lie bialgebra be coboundary. Then  $G$  contains a discrete subgroup  $\Gamma$  such that the factorgroup  $G/\Gamma$  is a Poisson Lie group, natural projection  $p : G \rightarrow G/\Gamma$  is a Poisson mapping, and  $G/\Gamma$  bears a global coordinate system in which its Poisson tensor is quadratic.*

**Proof** Let  $(\mathfrak{G}, \Delta_r)$ , where  $\Delta_r$  is as in Theorem 4, be a coboundary Lie bialgebra corresponding to the Poisson Lie structure on  $G$ . Then by Theorem 5 and Theorem 3  $\text{ad}_r^*/2$  is a quadratic Poisson bracket compatible with the associative algebra  $A$ . Let  $A_{inv}$  be a connected component of the unit in the group of all invertible elements of  $A$ . Then the bracket  $\text{ad}_r^*/2$  defines (by Theorem 6) a Poisson Lie structure on the group  $A_{inv}$  quadratic in the natural linear coordinates on this group.

A Lie algebra corresponding to  $A_{inv}$  is  $A_L = \mathfrak{G}$ . Since  $G$  is connected and simply connected, then  $A_{inv} = G/\Gamma$  for some discrete subgroup  $\Gamma \subset G$ , and the canonical homomorphism  $p : G \rightarrow G/\Gamma$  is a covering. It allows us to define a Poisson bracket  $\{\cdot, \cdot\}$  on  $G$  as an inverse image of the Poisson Lie bracket on  $A_{inv}$ . We will see now that this bracket is exactly the original one, which will complete the proof.

Define the mapping  $\Delta' : \mathfrak{G} \rightarrow \mathfrak{G} \wedge \mathfrak{G}$  by Drinfeld's recipe (see (21) above) using the bracket  $\{\cdot, \cdot\}$ . Since  $p$  is a covering, sufficiently small neighborhoods of units of the

group  $G$  and of the group  $G/\Gamma = A_{inv}$  are equivalent as Poisson manifolds. Thus  $\Delta'$  should be the same mapping as the above construction would give if applied to the group  $A_{inv}$ . Theorem 6 gives now that  $\Delta' = \Delta_r$ .

In particular,  $(\mathfrak{G}, \Delta') = (\mathfrak{G}, \Delta_r)$  is a Lie bialgebra and therefore by Theorem 5  $(G, \{\cdot, \cdot\})$  is a Poisson Lie group. Moreover, this Lie bialgebra coincides with the Lie bialgebra corresponding to the original Poisson Lie structure on  $G$ , and thus Theorem 5 shows that these two structures are the same. ■

Theorem 8 may fail to be true for a non-coboundary brackets — consider a linear bracket on the (additive) Lie group  $\mathbb{R}^n$ . Note also that the proof in fact gives more than the formulation promises: quadratic bracket on the group  $G/\Gamma$  is defined by an explicit (and easy) formula.

## 4 Examples

Besides those described here, several important examples can be found in the article [6].

**Example 1** Let  $A$  be an associative algebra with a solvable  $A_L$ . By classical Lie theorem any finite-dimensional linear representation of  $A_L$  has an eigenvector. Applying this to the regular representation, one obtains that  $A_L$  contains two elements,  $a$  and  $b$ , with  $[a, b] = sb$  for some constant  $s$ . The element  $r = a \otimes b - b \otimes a$  satisfies Yang–Baxter equation (1) and thus, according to Corollary 1, defines a quadratic Poisson bracket on  $A$ .

**Example 2** Consider a body  $\mathbf{H}$  of quaternions as a four-dimensional  $\mathbb{R}$ -algebra. Then any element  $r \in \mathbf{H} \wedge \mathbf{H}$  of the type

$$r = a \mathbf{i} \wedge \mathbf{j} + b \mathbf{i} \wedge \mathbf{k} + c \mathbf{j} \wedge \mathbf{k} \quad (25)$$

satisfies conditions of Theorem 3. Corresponding Poisson bracket on  $\mathbf{H}$  is:

$$\{x^1, x^2\} = x^2(bx^3 - ax^4) + c((x^3)^2 + (x^4)^2) \quad (26)$$

$$\{x^1, x^3\} = -x^3(cx^2 + ax^4) - b((x^2)^2 + (x^4)^2) \quad (27)$$

$$\{x^1, x^4\} = x^4(-cx^2 + bx^3) + a((x^2)^2 + (x^3)^2) \quad (28)$$

$$\{x^2, x^3\} = x^1(-bx^2 + cx^3) \quad (29)$$

$$\{x^2, x^4\} = -x^1(ax^2 + cx^4) \quad (30)$$

$$\{x^3, x^4\} = x^1(ax^3 - bx^4) \quad (31)$$

where  $x^1, x^2, x^3, x^4$  is a basis in  $\mathbf{H}^*$  dual to  $1, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbf{H}$ .

A simply connected Lie group corresponding to  $\mathbf{H}$  is a multiplicative group of all nonzero quaternions. It contains a subgroup  $\{z \in \mathbf{H}, \|z\| = 1\}$  isomorphic to  $\text{SU}(2)$ . It can be checked (see [10]) that this subgroup is a Poisson submanifold (and therefore a Poisson Lie group) with respect to all the brackets (26)–(31). Thus,  $\text{SU}(2)$  bears a three-parameter family of Poisson Lie structures.

The next example shows what can happen if the algebra  $A$  is associative but has no unit.

**Example 3** Consider a Lie algebra  $\mathfrak{G}$  satisfying an identity  $[[\mathfrak{G}, \mathfrak{G}], \mathfrak{G}] = 0$  (i.e.  $[[a, b], c] = 0$  for all  $a, b, c \in \mathfrak{G}$ ). A particular case here is Heisenberg algebra, a three-dimensional Lie algebra with generators  $p, q$ , and  $z$ , and relations  $[p, q] = \hbar z$ , and  $[z, p] = [z, q] = 0$ . Then  $a * b = \frac{1}{2}[a, b]$  is an associative operation with  $[a, b] = a * b - b * a$ . An arbitrary element  $r \in \mathfrak{G} \wedge \mathfrak{G}$  commutes with all the symmetric tensors from  $\mathfrak{G} \otimes \mathfrak{G}$  and therefore defines a zero Poisson bracket on  $\mathfrak{G}$ .

Consider now an algebra  $A = \langle 1 \rangle \oplus \mathfrak{G}$  where 1 is a formal unit. Then the simply connected Lie group  $G$  corresponding to  $\mathfrak{G}$  is an affine subspace  $1 + \mathfrak{G} \subset A$  (it is the usual implementation of  $G$  via exponents in the universal enveloping algebra of  $\mathfrak{G}$ ; the thing is that  $a^2 = 0$  for any  $a \in \mathfrak{G}$ ). Fix a basis  $\{e_i\} \in \mathfrak{G}$ ; it defines a natural coordinate system in the group  $G$ . The above construction applied to  $r = r^{ij}e_i \otimes e_j \in \mathfrak{G} \wedge \mathfrak{G} \subset A \wedge A$  gives now the following Poisson Lie bracket on  $G$ :

$$\{x^p, x^q\} = 2(r^{pl}a_{li}^q + r^{lq}a_{li}^p)x^i \quad (32)$$

where  $\{x^i\} \in \mathfrak{G}^*$  is a basis dual to  $\{e_i\}$ , and  $a_{ij}^k$  are structure constants of the associative operation  $*$  on  $\mathfrak{G}$ . Note that the bracket is nonzero (unlike bracket on  $\mathfrak{G}$  itself) and linear.

At last, give an example illustrating how important in Lemma 1 is the fact that  $\pi^{ij}(x)$  is smooth.

**Example 4** Consider the set  $\mathbb{R}^2$  with componentwise addition and multiplication. It is a two-dimensional  $\mathbb{R}$ -algebra with the unit  $u = (1, 1)$ . Then the Poisson bracket (singular)

$$\{f(x, y), g(x, y)\} = xy \log |x| (\partial f / \partial x \cdot \partial g / \partial y - \partial f / \partial y \cdot \partial g / \partial x) \quad (33)$$

is compatible with the multiplication in the algebra. The group of invertible elements here is  $\{(x, y) \mid x \neq 0 \neq y\}$ . Its connected component of a unit is  $\mathbb{R}_+^2$  where  $\mathbb{R}_+$  is a multiplicative group of positive real numbers. It is isomorphic to the *additive* group  $\mathbb{R}^2 = \{(\xi, \eta) \mid \xi, \eta \in \mathbb{R}\}$ . The isomorphism is given by

$$\xi = \exp(x), \quad \eta = \exp(y)$$

and maps bracket (33) to the linear Poisson bracket

$$\{f(\xi, \eta), g(\xi, \eta)\} = \xi(\partial f / \partial \xi \cdot \partial g / \partial \eta - \partial f / \partial \eta \cdot \partial g / \partial \xi). \quad (34)$$

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